THE BELL SYSTEM

TECHNICAL JOURNAL

DEVOTED TO THE SCIENTIFIC AND ENGINEERING ASPECTS OF ELECTRICAL COMMUNICATION

Volume 50 | December 1971 | Number 10

Statistics of a General Class of Avalanche Detectors With Applications to Optical Communication

By S. D. PERSONICK

(Manuscript received June 29, 1971)

Previous results on the statistics of avalanche detectors are generalized to the case where electrons and holes suffer collision ionizations with unequal probability. It is assumed here that the ratio of collision ionization probabilities per unit length of weaker-to-stronger carrier is a constant k independent of position in the high-field region. The moment-generating function of the random avalanche gain G is obtained as a function of k and the average gain $\bar{G}$, and is used to obtain Chernov bounds on error rates of digital optical receivers employing avalanche detectors. It is shown the required energy per pulse to achieve a given error rate decreases as k decreases for fixed $\bar{G}$. For each $k > 0$, there is an optimal mean gain $\bar{G}_{opt}$ resulting in minimum required energy per pulse. At $k = 0.1$, $\bar{G}_{opt} \approx 100$ and the required energy is within 10 dB of that required with very high gains (a few thousand) at $k = 0$.

I. INTRODUCTION

In a previous paper[1] results on the statistics of two particular avalanche detectors with applications to optical communication were

presented. It was required either that only one carrier suffer collision ionizations in the high-field region (unilateral gain) or that both carriers suffer collision ionizations with equal probability per unit length in the high-field region. The present work allows for more general unequal ionization probabilities per unit length with the requirement only that the ratio of the two quantities be constant throughout the high-field region. The moment-generating function of the random gain is obtained as a function of this ratio and the average gain. The results are consistent with unpublished conjectures of R. J. McIntyre.[2] The moment-generating functions are used to obtain Chernov bounds on the error rates of digital optical receivers employing avalanche detectors and using either coherent or incoherent light. Results on avalanche statistics are summarized in Section V. Numerical results on the Chernov bounds are given in Section VI (6.5) and Section VII.

II. MODEL OF THE AVALANCHE DETECTOR

The avalanche detector is a device in which thermally or optically generated hole-electron pairs generate additional hole-electron pairs through collision ionizations. Within the device there is a "high-field region" where holes have probability $\beta(x)$ per unit length (which depends

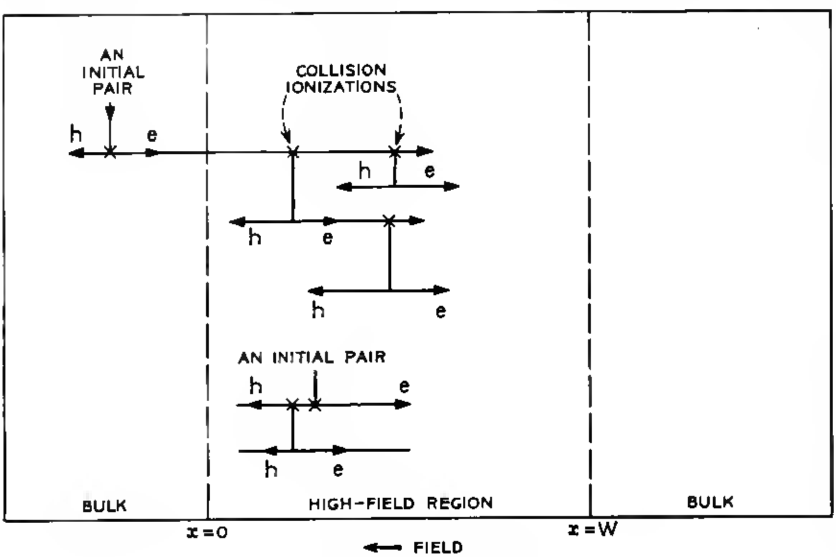

Fig. 1—Avalanche detector.

upon the position x) of suffering a collision ionization as they travel to the left under the influence of the electron field (see Fig. 1). Electrons traveling to the right have a probability $\alpha(x)$ per unit length of collision ionization. Carriers can be created within the high-field region due to thermal effects or due to the presence of incident light. Carriers can also drift into the region if they are generated outside of the region. It is assumed that all collision ionizations are independent. This requires that the mean distance between ionizing collisions be large compared to the distance over which a carrier can randomize its momentum after a collision.[3] Hole-electron pairs created through collision ionization can in turn generate additional pairs by the same mechanism. This is the avalanche process.

III. THE STATISTICS

We seek the statistics of the random total number of hole-electron pairs which result ultimately through collision ionizations when an initial hole-electron pair is injected into the high field at some position x. Define $p_g(n, x)$ as the probability that n pairs ultimately result including the initially injected pair. The moment-generating function of the number of pairs $M_g(s)$ is therefore[4]

$$M_g(s, x) = \sum_{1}^{\infty} p_g(n, x)e^{sn}. \tag{1}$$

We shall derive $M_g(s, x)$. Before proceeding we must review some well known results which will be needed in that derivation.

If $\{x_i\}$ are random variables which are independent, then the moment-generating function of the sum of the $\{x_i\}$ is the product of the individual moment-generating functions.[4] The semi-invariant moment-generating function SIMGF of a random variable X having probability density $p_X(x)$ is the natural logarithm of the moment-generating function of X

$$\psi_X(s) \equiv \ln [M_X(s)] = \ln \left[\sum_{-\infty}^{\infty} p_X(x)e^{sx} \right]. \tag{2}$$

The SIMGF of a sum of n independent random variables is the sum of the individual SIMGF's.

We can now proceed to derive $M_g(s, x)$. Divide the high-field region into K intervals of width $dX = W/K$. See Fig. 2. Label these intervals $1, 2, 3, \cdots j, \cdots K$. If a hole-electron pair is injected into interval j, define, as above, the probability density of the total number of pairs ultimately resulting in the avalanche process (including the initial pair)

as $p_0(n, x)$ where x is taken as the center of interval j. The hole of the initial pair moves to the left and the electron to the right toward $x = 0$ or $x = W$ respectively. As they pass through their respective intervals, new pairs may be created in each through collision ionization. We shall assume that the interval width dX is sufficiently narrow so that the initial pair carriers create either one or no new pairs in each interval. If the initial hole or electron generates a new pair in some interval k, then that new pair will ultimately generate N_k pairs including itself through the avalanche process. Thus with each *interval* we can associate a number of pairs N_k. This number equals zero if the appropriate initial pair carrier suffers no collision ionizations in interval k. This number equals one or more if the appropriate initial pair carrier suffers a collision ionization in interval k. The total number of pairs ultimately generated through the avalanche process including the initial pair is one plus the sum of the $\{N_k\}$. Since collision ionizations are all independent, all the N_k are independent. Thus we have the SIMGF of the total number of pairs given by

$$\psi_0(s, x) = s + \sum_{k=1}^{W/dX} \psi_{N_k}(s) \tag{3}$$

where s is the SIMGF of the deterministic initial pair and $\psi_{N_k}(s)$ is the SIMGF of N_k.

The SIMGF of N_k is obtained as follows. The probability that

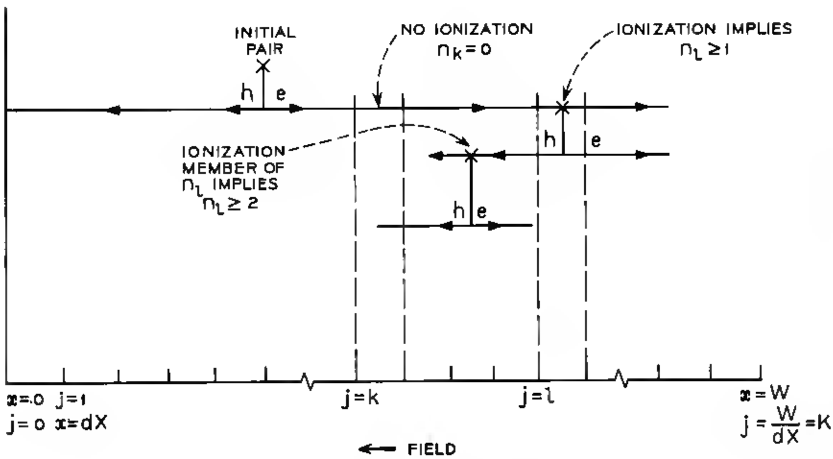

Fig. 2—Avalanche process.

$N_k = 0$ is $1 - \gamma\, dX$ where $\gamma = \alpha(x)$ if interval k is to the right of interval j where the initial pair enters, $\gamma = \beta(x)$ if k is to the left of j. (x is evaluated at the center of interval k.) The probability that $N_k = Z > 0$ is the probability of a collision ionization in interval k times the probability that a new pair created at interval k ultimately results in Z pairs including itself. That is

$$\Pr\,(N_k = Z) = \gamma\, dX\; p_o(Z, x) \quad \text{for} \quad Z > 0 \tag{4}$$

with x evaluated at the center of interval k. Thus

$$\begin{aligned}
\psi_{N_k}(s) &= \ln\left[\sum_{Z=0}^{\infty} \Pr\,(N_k = Z)e^{sZ}\right] \\
&= \ln\left[(1 - \gamma\, dX)e^{s0} + \sum_{Z=1}^{\infty} \gamma\, dX\, p_o(Z, x)e^{sZ}\right] \\
&= \ln\,[1 - \gamma\, dX + \gamma\, dX\, M_o(s, x)] \\
&= \ln\,[1 - \gamma\, dX + \gamma\, dX\, e^{\psi_o(s,x)}]
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
\gamma &= \alpha(x) \quad \text{if} \quad k > j \\
\gamma &= \beta(x) \quad \text{if} \quad k < j \\
x &= \text{center of interval } k.
\end{aligned}$$

Using (3) and (5) and taking the limit as dX gets infinitely small, one obtains

$$\psi_o(s, x) = s + \int_0^x \beta(x')[e^{\psi_o(s,x')} - 1]\, dx' + \int_x^W \alpha(x')(e^{\psi_o(s,x')} - 1)\, dx'. \tag{6}$$

Equation (6) is the critical equation for determining $\psi_o(s, x)$, and thus $M_o(s, x) = \exp\,[\psi_o(s, x)]$. Using Leibnitz's rule for differentiation of integrals one obtains

$$\frac{\partial}{\partial x}\psi_o(s, x) = [\beta(x) - \alpha(x)](e^{\psi_o(s,x)} - 1). \tag{7}$$

The solution of (7) is

$$\psi_o(s, x) = \ln\left[\frac{1}{1 - C\exp\left(\int_0^x [\beta(x') - \alpha(x')]\, dx'\right)}\right] \tag{8}$$

where $C = (e^{\psi_e(s,0)} - 1)/e^{\psi_e(s,0)}$ which can be checked by substitution.

Substituting (8) into (6) one obtains the particular result

$$\psi_e(s, 0) = s + \int_0^W \alpha(x')\left[\frac{1}{1 - C\exp\left(\int_0^{x'} [\beta(x'') - \alpha(x'')]\,dx''\right)} - 1\right] dx'. \quad (9)$$

If one makes the *assumption*[3] that at each point in the high-field region

$$\beta(x) = k\cdot\alpha(x) \quad (10)$$

where k is a constant, one can solve (9) to obtain

$$\psi_e(s, 0) = s - \delta + \frac{1}{k-1}\ln\left[\frac{e^{(k-1)\delta}}{M_e(s, 0) - e^{(k-1)\delta}[M_e(s, 0) - 1]}\right]$$

$$= s + \frac{1}{1-k}\ln\left[M_e(s, 0) - e^{(k-1)\delta}[M_e(s, 0) - 1]\right] \quad (11)$$

where

$$M_e(s, 0) = e^{\psi_e(s,0)} \quad \text{and} \quad \delta = \int_0^W \alpha(x)\,dx.$$

One can write (11) in another way by making a substitution.* Define $\delta'(s)$ implicitly by

$$e^{\delta'(s)} = M_e(s, 0)e^{\delta - s}, \quad (12)$$

that is,

$$e^{\psi_e(s,0)} = M_e(s, 0) = e^{s - \delta + \delta'(s)}. \quad (13)$$

Using (12) in (11) one obtains the implicit equation

$$e^{-k\delta'(s)} - e^{-\delta'(s)} = e^{s}[e^{-k\delta} - e^{-\delta}] \quad (14)$$

which determines $\delta'(s)$ and thus $M_e(s, 0)$ through (13).

Equation (14) is still not explicit. A numerical technique for solution is discussed in the next section. One can use (11) [or (14)] and (8) to obtain $\psi_e(s, x)$ or $M_e(s, x)$ for any x. Recall that x is the point of entry of the initial pair. In the applications we shall be concerned with $x = 0$ or $x = W$. That is, pairs are generated in a drift region outside the high-field region with carriers drifting into the high-field region.

* Equation (14) will follow from (11) and (12) by tedious algebra. Further, (14) will not be used in the following results except to compare with McIntyre's work in Appendix A.

The equation (14) is consistent with some unpublished conjectures of McIntyre given in Appendix A.

IV. NUMERICAL SOLUTIONS

Equations (11) through(14) can be solved numerically. One technique is to differentiate (11) to obtain the result

$$\frac{\partial}{\partial s} M_g(s, 0) = M_g(s, 0)\left(\frac{k-1}{k}\right)\left[1 - \frac{1}{k}\,(M_g(s, 0)e^{-s}e^{\delta})^{k-1}\right]^{-1} \tag{15}$$

where $M_g(0, 0) = 1$.

Equation (15) can be integrated with a computer to obtain $M_g(s, 0)$ explicitly.

V. SUMMARY OF ANALYTIC RESULTS ON AVALANCHE STATISTICS

From Sections I through IV and a previous paper by this author,[1] we obtained the following:

Assumptions:

Holes travel toward $x = 0$, electrons toward $x = W$.
Hole ionization probability per unit length $= \beta(x)$.
Electron ionization probability per unit length $= \alpha(x)$.
$\beta(x) = k \cdot \alpha(x)$, k a constant for all x.
High-field region width $= W$.

Definitions:

$$\delta \equiv \int_0^W \alpha(x)\, dx,$$

$p_g(n, x) =$ probability that if an initial pair enters the high-field region at point x, n pairs will ultimately result through the avalanche process including the initial pair,

$$\bar{G}(x) = \text{mean avalanche gain} = \sum_1^\infty n p_g(n, x),$$

$$M_g(s, x) \equiv \text{moment-generating function of } p_g(n, x)$$

$$= \sum_{n=1}^\infty p_g(n, x)e^{sn}.$$

Results:

$$1.\ M_g(s, x) = \left[1 - \frac{(M_g(s, 0) - 1)}{M_g(s, 0)} \exp\left[(k - 1)\int_0^x \alpha(x')\,dx'\right]\right]^{-1} \quad \text{for all} \quad k. \tag{16}$$

For $k = 0$ (Unilateral Gain)

$$2a.\ M_g(s, 0) = [1 - e^{\delta}[1 - e^{-s}]]^{-1} \quad \text{where} \quad \bar{G}(0) = e^{\delta}. \tag{17}$$

$$2b.\qquad p_g(n, 0) = \frac{1}{G}\left(\frac{G - 1}{G}\right)^{n-1} \quad \text{where} \quad G = \bar{G}(0). \tag{18}$$

For $k \neq 0,\ k \neq 1$

$$3.\ \frac{\partial}{\partial s} M_g(s, 0) = M_g(s, 0)\left(\frac{k - 1}{k}\right)\left[1 - \frac{1}{k}(M_g(s, 0)e^{-s}e^{\delta})^{k-1}\right]^{-1},$$

where

$$M_g(0, 0) = 1 \tag{19}$$

where

$$\bar{G}(0) = \left(\frac{k - 1}{k}\right)\left[1 - \frac{1}{k} e^{\delta[k-1]}\right]^{-1}.$$

For $k = 1$ (Equal Ionization)

$$4.\qquad \frac{\partial}{\partial s} M_g(s, 0) = M_g(s, 0)\left[1 - \left(\frac{G - 1}{G}\right)M_g(s, 0)\right]^{-1}$$

$$M_g(0, 0) = 1$$

where

$$G = \bar{G}(0). \tag{20}$$

VI. APPLICATIONS TO RECEIVERS USING AVALANCHE DETECTORS

6.1 *General Comments*

We shall next apply the results of Section V to obtain bounds on the error rates of digital receivers. The receivers to be discussed here are the single- and twin-channel systems described below. We shall upper-bound the power required at the receiver to obtain a desired error rate using the Chernov bounds.

6.2 *The Receivers*

The twin-channel receiver is shown in Fig. 3. Depending on the state of a binary information source, one of two channels has optical output

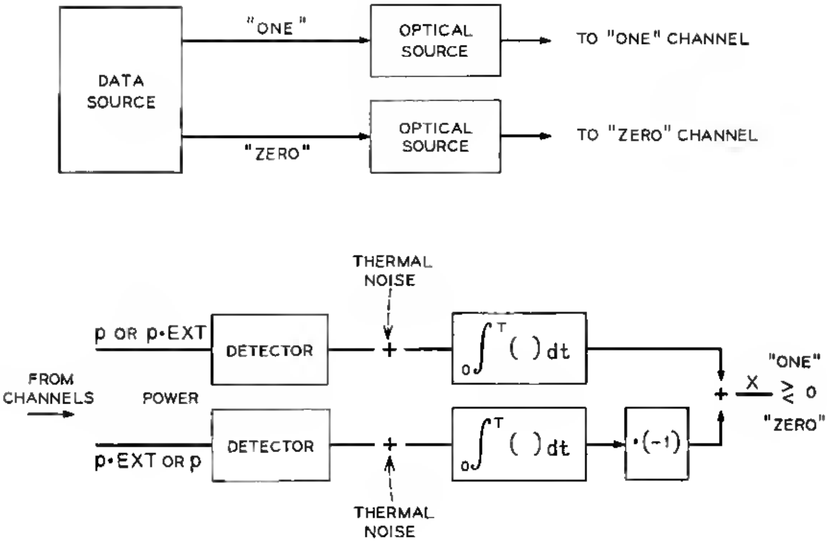

Fig. 3—Twin-channel system.

power p for duration T while the other has output power $p \cdot EXT$ for duration T. EXT signifies extinction. EXT would ideally be zero, but is left finite and less than unity for practical reasons. The optical power falling on an avalanche photo diode causes the emission of photoelectrons which are multiplied along with the detector "dark current" through the avalanche gain mechanism. The detector outputs (multiplied counts) are integrated in devices having thermal noises referred to their respective inputs. The integrator outputs are subtracted and the difference X is compared to a threshold of zero to decide what the information state was.

The single-channel system is essentially half the twin-channel system as shown in Fig. 4. The single integrator output X is compared to a threshold γ to decide upon the information state.

6.3 *The Chernov Bounds*

The Chernov bound is a useful tool for bounding the probability that a random variable X will lie above or below a given threshold γ. It is given as follows[5]

$$\Pr(X > \gamma) \leqq e^{\psi_X(s) - s\dot{\psi}_X(s)} \Big|_{\gamma = \dot{\psi}_X(s)}$$

provided $s > 0$,

$$\Pr(X < \gamma) \leqq e^{\psi_X(s) - s\psi'_X(s)} \Big|_{\gamma = \psi'_X(s)}$$

$$\text{provided } s < 0, \qquad (21)$$

where

$$\psi_X(s) = \text{SIMGF of } X \equiv \ln \left[\int_{-\infty}^{\infty} p_X(x) e^{sx} \, dx \right]$$

and

$$\psi'_X(s) = \frac{\partial}{\partial s} \psi_X(s).$$

6.4 *Chernov Bounds for the Two Systems*

6.4.1 *Preliminaries*

We wish to determine the required power p to achieve a desired decision error probability for each of the receivers discussed in Section 6.2 with various types of detectors and various values of other system parameters such as dark current and thermal noise in the integrators. To apply the Chernov bounds we will need the SIMGF's of the variables X at the outputs of the receivers. (See Figs. 3 and 4.)

An important result needed here is the following:

Lemma: If C is integer-valued and greater than or equal to zero; and if $U = \sum_0^C g_i$ where the g_i are independent, identically distributed random variables (that is, each "count" produced by the C process independently

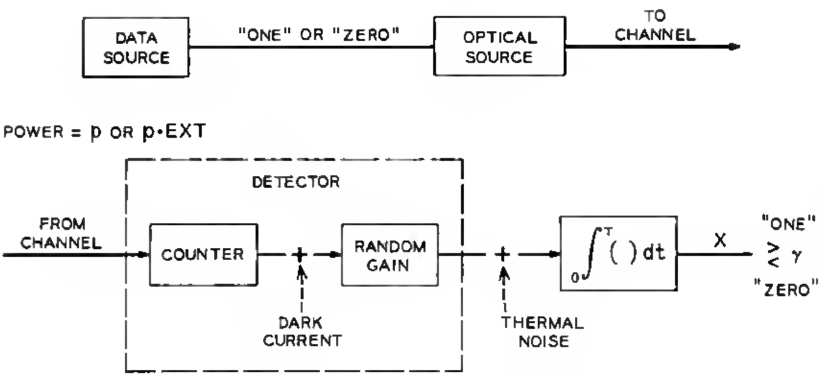

Fig. 4—Single-channel system.

generates g_i contributions to the U process through some gain mechanism); then the SIMGF of U is[1]

$$\psi_U(s) = \psi_C(\psi_g(s)) \tag{22}$$

where $\psi_g(s)$ is the SIMGF of the identically distributed random variables g_i.

One can model an avalanche photodiode as a photon counter followed by a random avalanche multiplier. Each photon counter output "count" produces a random number of "counts" at the multiplier output. Thus, since we know from Section V the SIMGF of a random multiplier which corresponds to $\psi_g(s)$ in (22), and since we seek the SIMGF of the photodiode output which corresponds to $\psi_U(s)$ in (22), it follows that we need the SIMGF of the number of counts emitted by a photon counter with light incident upon it, which corresponds to $\psi_C(s)$ in (22).

6.4.2 *Photon Counter Statistics*

6.4.2.1 *Coherent Light.* If the light incident upon the photon counter is coherent, then it is well known that the SIMGF of the total counts emitted in an interval T is given by[6] (see Appendix B)

$$\psi_C(s) = [LAMSIG + LAMD]\,[e^s - 1] \tag{23}$$

where

$LAMSIG$ = Total incident light energy $\cdot\, \eta/\hbar\Omega$
η = Detector quantum efficiency
$\hbar\Omega$ = Energy of photon at optical frequency used
$LAMD$ = Mean number of dark current counts before avalanche gain in an interval T.

6.4.2.2 *Incoherent Light.* If the incident light is incoherent with H independent spatial-temporal "degrees of freedom," then the SIMGF of the total counts emitted by the photon counter in interval T is (see Appendix B)

$$\psi_C = LAMD[e^s - 1] + \ln\left[\left[1 - \frac{LAMSIG}{H}(e^s - 1)\right]^{-H}\right] \tag{24}$$

where $LAMSIG$ is the average total incident energy times $\eta/\hbar\Omega$. Clearly (24) is the same as (23) as H approaches infinity, which is a well known result.

6.4.3 *Final Calculations*

6.4.3.1 *Twin-Channel System.* Since, for the twin-channel receiver, X consists of the difference of two random integrator outputs, we need

the following well known result.[4] If $X = X_1 - X_2$, and if X_1 and X_2 are independent, then the SIMGF of X is

$$\psi_X(s) = \psi_{X_1}(s) + \psi_{X_2}(-s). \tag{25}$$

Each integrator output contains the sum of the counts emitted by its detector and the integral of its thermal noise. The SIMGF of the random variable N obtained when Gaussian thermal noise spectral height N_0 is integrated over an interval T is well known to be

$$\psi_N(s) = \frac{s^2}{2} N_0 T. \tag{26}$$

Using (22) through (26) we obtain for the SIMGF of the twin-channel receiver output X when the information is in state "one", and the optical source is coherent,

$$\psi_X(s) = s^2 N_0 T + \left[\frac{p \cdot T \cdot \eta}{\hbar\Omega} + LAMD\right][M_g(s) - 1] + \left[\frac{p \cdot T \cdot EXT \cdot \eta}{\hbar\Omega} + LAMD\right][M_g(-s) - 1] \tag{27}$$

where

$LAMD$ = mean number of dark current counts before avalanche gain in an interval T

N_0 = spectral height of the thermal noises referred to the integrator inputs.

$M_g(s)$ is obtained from (16) through (20) depending upon the particular gain mechanism. If the optical source is incoherent, we have

$$\psi_X(s) = s^2 N_0 T + LAMD[M_g(s) + M_g(-s) - 2] + \ln\left[\left[1 - \frac{p \cdot T \cdot \eta}{H\hbar\Omega}[M_g(s) - 1]\right]^{-H}\right] + \ln\left[\left[1 - \frac{p \cdot T \cdot EXT \cdot \eta}{H\hbar\Omega}[M_g(-s) - 1]\right]^{-H}\right]. \tag{28}$$

We seek the probability that when the information is in state "one", X is less than zero, and we therefore decide that the information state was "zero." That is, we seek the error probability. One can use (27) or (28) and the Chernov bound of (21) to determine the required value of $LAMSIG \equiv p \cdot T \cdot \eta/(\hbar\Omega)$ to achieve a desired error probability. Since the twin-channel receiver is symmetric, the error probability

when the information is in state "zero" is the same as when it is in state "one."

6.4.3.2 *Single-Channel System.* For the single-channel receiver, we need the SIMGF of X under both information states. Call X_1 the random variable X when the information is in the state "one." Call X_0 the random variable X when the information is in state zero. Using results of Section 6.4, one obtains for coherent light

$$\psi_{X_1}(s) = \frac{s^2 N_0 T}{2} + \left[LAMD + \frac{p \cdot T \cdot \eta}{\hbar\Omega}\right][M_g(s) - 1]$$

$$\psi_{X_0}(s) = \frac{s^2 N_0}{2} + \left[LAMD + \frac{p \cdot T \cdot EXT \cdot \eta}{\hbar\Omega}\right][M_g(s) - 1]. \tag{29}$$

For incoherent light

$$\psi_{X_1}(s) = \frac{s^2 N_0 T}{2} + LAMD[M_g(s) - 1] + \ln\left[\left[1 - \frac{p \cdot T \cdot \eta}{H\hbar\Omega}[M_g(s) - 1]\right]^{-H}\right]$$

$$\psi_{X_0}(s) = \frac{s^2 N_0 T}{2} + LAMD[M_g(s) - 1] + \ln\left[\left[1 - \frac{p \cdot T \cdot EXT \cdot \eta}{H \cdot \hbar\Omega}[M_g(s) - 1]\right]^{-H}\right]. \tag{30}$$

One can then use the results of (29) and (30) along with the Chernov bounds of (21) to simultaneously find values of $LAMSIG = p \cdot T \cdot \eta/(\hbar\Omega)$ and the threshold γ (see Fig. 4) to ensure some desired error probability (which for convenience here will be the same for either information state).

6.5 *Numerical Results*

The Chernov bounds described above were evaluated numerically. The results are displayed on the attached figures described below. The range of parameter values is realistic and practical, to the best of this author's knowledge. The curves presented are those deemed most interesting by the author. Other calculations can of course be made. Parameters used are defined as follows:*

$LAMSIG$ = Required mean number of detected photons per pulse in the "on" channel of the twin-channel

* SIG, EXT, G, K, H, and $LAMD$ are input parameters to the program which calculates $LAMSIG$ for a desired error rate.

receiver or in the "one" state of the single-channel receiver.

$LAMSIG \cdot EXT$ = Mean number of detected counts per pulse in the "off" channel of the twin-channel receiver or in the "zero" state of the single-channel receiver.

SIG = Normalized thermal noise standard deviation

$$= \{4k\theta T/[\mathrm{Re}^2]\}^{\frac{1}{2}} = \{4k\theta C/e^2\}^{\frac{1}{2}}$$

where e = electron charge, $k\theta$ = Boltzmann's constant·absolute temperature, R = equivalent noise resistance at integrator input, T = pulse duration, $C = T/R$ = integrator equivalent input capacitance. For the results to follow, a reasonable value of SIG was chosen to be 6000.

G = Mean avalanche gain.

H = Temporal-spatial diversity for incoherent carrier case.

k = Ratio of ionization probability per unit length of weaker and stronger ionizing carriers.*

$LAMD$ = Dark current counts per interval T before avalanche gain.

Fig. 5

$LAMSIG$ vs G is plotted for the twin-channel case with k as parameter. SIG was set at 6000, the error rate is 10^{-9}, $LAMD$ was set at 5 counts and $EXT = 0.01$. $H = 10{,}000$ which is equivalent to assuming a coherent carrier.

Fig. 6

The value at optimal gain of $LAMSIG$ vs k is plotted. Points are tagged with the optimal G. The receiver is a twin-channel system with $SIG = 6000$, $EXT = 0.01$, $LAMD = 5$. H is 10,000 which is equivalent to assuming a coherent carrier. The error rate is 10^{-9}.

Fig. 7

$LAMSIG$ vs G is plotted for two values of error rate 10^{-9} and 10^{-5} for

* For these calculations it was assumed that the detector is designed so that the stronger ionizing carriers generated optically or associated with dark current enter the high-field region from a drift region outside the high-field region. This corresponds to initial pairs entering the gain mechanism of $x = 0$ or $x = W$ as discussed in Section III.

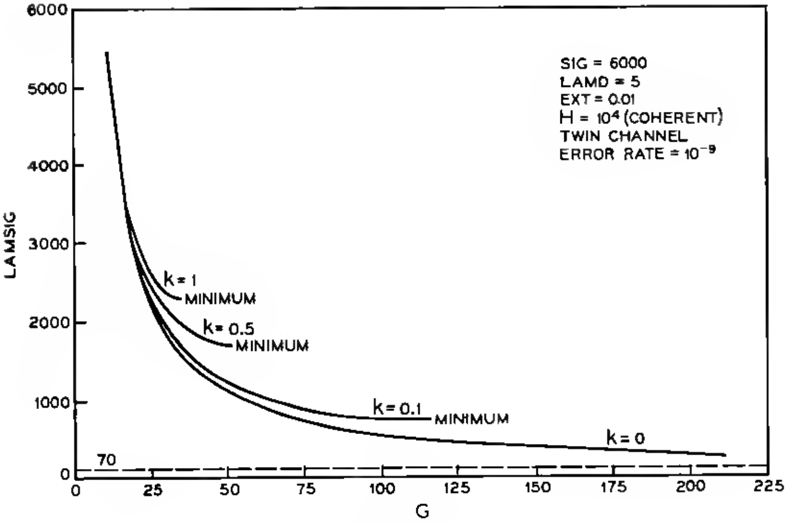

Fig. 5—LAMSIG versus gain.

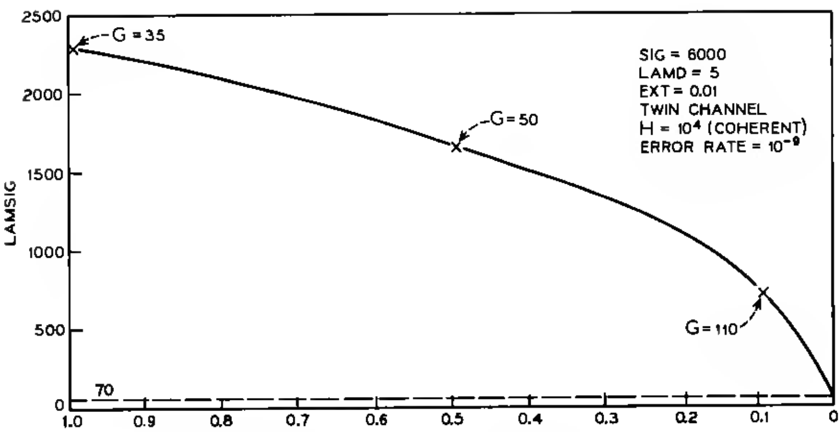

Fig. 6—LAMSIG at optimal gain versus k.

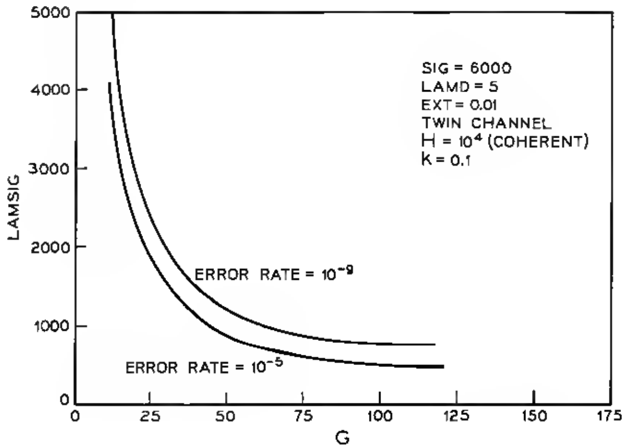

Fig. 7—LAMSIG versus gain.

a twin-channel system with $SIG = 6000$, $EXT = 0.01$, $H = 10{,}000$ $LAMD = 5$, $k = 0.1$.

Fig. 8

Single- and twin-channel systems are compared. $LAMSIG$ vs G is plotted for $SIG = 6000$, $EXT = 0.01$, $LAMD = 5$, $H = 10{,}000$, error rate $= 10^{-9}$, $k = 0$. Note that from an average power viewpoint the single-channel system is 3 dB better than shown if the binary information source is random, since $LAMSIG$ is the energy in "one" state.

Fig. 9

Same as Fig. 8 except $k = 1$. Note the scale change.

Fig. 10

$LAMSIG$ vs G for $H = 100$ and $H = 10{,}000$ for twin-channel system. $SIG = 6000$, $EXT = 0.01$, $LAMD = 5$, $k = 0$, error rate $= 10^{-9}$.

6.6 *Further Comments*

When systems were investigated for sensitivity to the choice $LAMD = 5$, $EXT = 0.01$, it was found that insignificant changes in $LAMSIG$ vs G occurred when various combinations of $LAMD = 5$ or 50, $EXT = 0.01$ or 0.001 were tried.

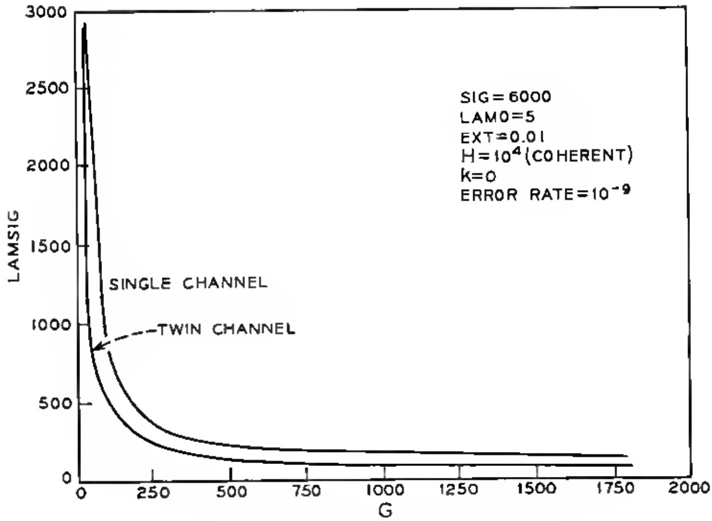

Fig. 8—LAMSIG versus gain.

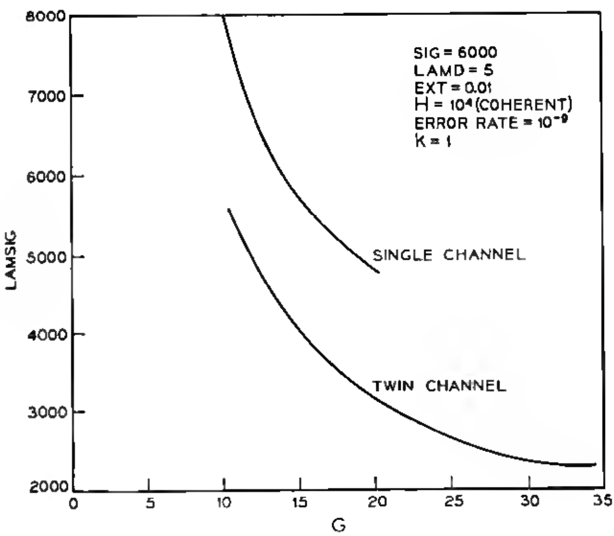

Fig. 9—LAMSIG versus gain.

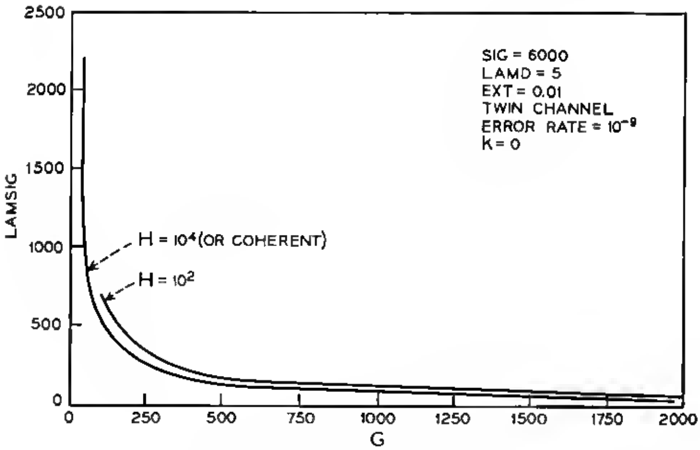

Fig. 10—LAMSIG versus gain.

VII. CONCLUSIONS ON APPLICATIONS

If one assumes that the Chernov bounds are sufficiently tight so that actual energy required per bit to achieve specified error rates can be compared for various system parameters by comparing the bounds,* then one can conclude the following.

(*i*) Define k as the ratio of the collision ionization probabilities per unit length of the weaker-ionizing to the stronger-ionizing carrier (carriers are of course holes and electrons). Assume that the detector is designed "well" such that optically and thermally generated carriers enter the high-field region from a drift region outside. From the bounds, one obtains the result that the required energy per pulse to achieve a desired error rate decreases as k decreases for fixed average avalanche gain. A value $k = 0$ is best; but a value $k = 0.1$ will allow one to operate with energy within 10 dB of that required at very high gains with a $k = 0$ device. For each value of k except zero there is an optimal gain resulting in minimum required energy per pulse. The optimal gain is larger for smaller k. At $k = 0.1$, the optimal gain is about 100. At $k = 0$, the optimal gain is infinite, but a gain of a few thousand allows

* For simple cases where both the bounds and actual energy requirements can be obtained (for instance for the $k = 0$ case) the two results differ by a few dB or less.

close to optimal required energy per pulse. One can conclude that a silicon device with $k = 0.1$ and a gain of about 100 would be a good choice for an optical detector. This is true since a detector with k less than 0.1 and yet having gain significantly higher than 100 is not available at this time.

(*ii*) The required energy per pulse for systems using incoherent optical sources differs from that for systems using coherent sources by less than a few dB provided the product of the source bandwidth and the pulse duration exceeds 100. This is true even if there is no spatial incoherence of the light at the detector.

(*iii*) For reasonable parameter values, and assuming a random information stream, the single-channel receiver requires about 1.5 dB less energy per pulse to achieve a desired error rate than the twin-channel receiver.

(*iv*) The required energy per pulse is insensitive to reasonable values of dark current and extinction ratios.

(*v*) For a particular system, a change in the desired error rate from 10^{-9} to 10^{-5} results in a change in the required energy per pulse of 1 to 3 dB, depending upon the avalanche gain. This shows that the required energy per pulse is fairly insensitive to the error rate. On the other hand, this means that poor error rates will result if insufficient loss margin is provided. That is, a small lowering of the received energy can greatly increase the error rate.

APPENDIX A

In an unpublished work, McIntyre conjectures (from special case calculations) that the probability density of the random gain, defined here as $p_g(n, 0)$ is given by

$$p_g(n, 0) = \frac{\Gamma\left(\dfrac{n}{1-k} + 1\right) e^{-\delta}(e^{-k\delta} - e^{-\delta})^{n-1}}{n!\,\Gamma\left(\dfrac{n}{1-k} + 2 - n\right)}$$

where k and δ are the same as in (10) through (13).

If one makes the assumption that the conjectured $p_g(n, 0)$ has sum over n normalized to unity for each value of k and for each δ, then one obtains the result of (14) by using the definition of the moment-generating function and the normalization property.

APPENDIX B

If light of *known* intensity falls upon a photon counter during an interval T, then the probability density of the total number of counts emitted is well known[6] to be Poisson distributed as follows

$$p_C(n) = [\Lambda + LAMD]^n \frac{e^{-(LAMD+\Lambda)}}{n!}. \tag{31}$$

Where Λ* is the total energy incident in the interval T times $\eta/\hbar\Omega$, $LAMD$ is the mean number of dark current counts per second times the interval T, and $\eta/\hbar\Omega$ is the detector quantum efficiency divided by the energy in a photon.

The moment-generating function of the distribution of (31) is given by

$$M_C(s) = \exp [[\Lambda + LAMD][e^s - 1]]. \tag{32}$$

If the incident light is a stochastic process, then the moment-generating function of the output count distribution is obtained by averaging (32) over the probability density of the stochastic total energy incident in the interval T

$$M_C(s) = \int_0^\infty \exp ([\Lambda + LAMD][e^s - 1])p(\Lambda)\, d\Lambda. \tag{33}$$

An incoherent light field is normally taken to mean that the complex envelope of the classical field is a complex Gaussian random process. That is, such a field incident on the photon counter plane can be written as

$$\begin{aligned} E(\rho, t) &= \sqrt{2}\, \mathrm{re}\{\epsilon(\rho, t)e^{i\Omega t}\} \\ &\rho \,\varepsilon \text{ counter plane} \\ &t \,\varepsilon\, (0, T) \end{aligned} \tag{34}$$

where $\epsilon(\rho, t)$ is a complex Gaussian random process.

If one expands $\epsilon(\rho, t)$ in its Karhunen–Loeve eigenfunctions,[5] one obtains

$$\begin{aligned} \epsilon(\rho, t) &= \sum e_k\phi_k(\rho, t) \\ &\rho \,\varepsilon \text{ counter plane} \\ &t \,\varepsilon\, (0, T) \end{aligned} \tag{35}$$

* In the text, Λ is called $LAMSIG$.

where

$$\int_{\text{counter plane}} \int_0^T \phi_k(\rho, t)\phi_i^*(\rho, t)\, d^2\rho\, dt = \delta_{k,i}$$

and the coefficients e_k are independent complex Gaussian random variables satisfying

$$\begin{aligned} \langle e_k e_i^* \rangle &= \gamma_k \delta_{k,i} \\ \langle e_k e_i \rangle &= 0. \end{aligned} \tag{36}$$

The energy incident upon the photon counter is

$$\frac{\hbar\Omega}{\eta}\Lambda = \int \epsilon(\rho, t)\epsilon^*(\rho, t)\, d^2\rho\, dt = \sum |e_k|^2. \tag{37}$$

If one assumes an equal distribution of average energy in roughly H "modes,"

$$\begin{aligned} \gamma_k &= \gamma, \qquad 1 \leqq k \leqq H \\ &= 0, \qquad k > H \end{aligned} \tag{38}$$

then it follows that from (33) and the complex Gaussian statistics of the e_k that

$$M_C(s) = \exp\,[LAMD(e^s - 1)] \cdot \left[1 - \frac{\eta}{\hbar\Omega}\gamma(e^s - 1)\right]^{-H}. \tag{39}$$

Assumption (38) implies that the energy of the incoherent light is roughly equally distributed in H degrees of freedom.

REFERENCES

1. Personick, S. D., "New Results on Avalanche Multiplication Statistics with Applications to Optical Detection," BSTJ, *50*, No. 1 (January 1971), pp. 167–189.
2. McIntyre, R. J., unpublished work (see Appendix A).
3. McIntyre, R. J., "Multiplication Noise in Uniform Avalanche Diodes," IEEE Trans. Elec. Dev., *ED-13*, No. 1 (January 1966), pp. 164–168.
4. Davenport, W., and Root, W., *Random Signals and Noise*, New York: John Wiley & Sons, 1958.
5. Van Trees, H. L., *Detection Estimation and Modulation*, *I*, New York: John Wiley & Sons, 1967.
6. Glauber, R. J., *Quantum Optics and Electronics*, *Les Honches 1964*, Lecture Notes at Session of the Summer School of Theoretical Physics at the Univ. of Grenoble, Ed. De Witt, New York: Gordon and Breach, 1965, p. 64.

THE BELL SYSTEM TECHNICAL JOURNAL

DEVOTED TO THE SCIENTIFIC AND ENGINEERING ASPECTS OF ELECTRICAL COMMUNICATION

Volume 50 | December 1971 | Number 10

Statistics of a General Class of Avalanche Detectors With Applications to Optical Communication

By S. D. PERSONICK

(Manuscript received June 29, 1971)

Previous results on the statistics of avalanche detectors are generalized to the case where electrons and holes suffer collision ionizations with unequal probability. It is assumed here that the ratio of collision ionization probabilities per unit length of weaker-to-stronger carrier is a constant k independent of position in the high-field region. The moment-generating function of the random avalanche gain G is obtained as a function of k and the average gain $\bar{G}$, and is used to obtain Chernov bounds on error rates of digital optical receivers employing avalanche detectors. It is shown the required energy per pulse to achieve a given error rate decreases as k decreases for fixed $\bar{G}$. For each $k > 0$, there is an optimal mean gain $\bar{G}_{opt}$ resulting in minimum required energy per pulse. At $k = 0.1$, $\bar{G}_{opt} \approx 100$ and the required energy is within 10 dB of that required with very high gains (a few thousand) at $k = 0$.

I. INTRODUCTION

In a previous paper[1] results on the statistics of two particular avalanche detectors with applications to optical communication were

presented. It was required either that only one carrier suffer collision ionizations in the high-field region (unilateral gain) or that both carriers suffer collision ionizations with equal probability per unit length in the high-field region. The present work allows for more general unequal ionization probabilities per unit length with the requirement only that the ratio of the two quantities be constant throughout the high-field region. The moment-generating function of the random gain is obtained as a function of this ratio and the average gain. The results are consistent with unpublished conjectures of R. J. McIntyre.[2] The moment-generating functions are used to obtain Chernov bounds on the error rates of digital optical receivers employing avalanche detectors and using either coherent or incoherent light. Results on avalanche statistics are summarized in Section V. Numerical results on the Chernov bounds are given in Section VI (6.5) and Section VII.

II. MODEL OF THE AVALANCHE DETECTOR

The avalanche detector is a device in which thermally or optically generated hole-electron pairs generate additional hole-electron pairs through collision ionizations. Within the device there is a "high-field region" where holes have probability $\beta(x)$ per unit length (which depends

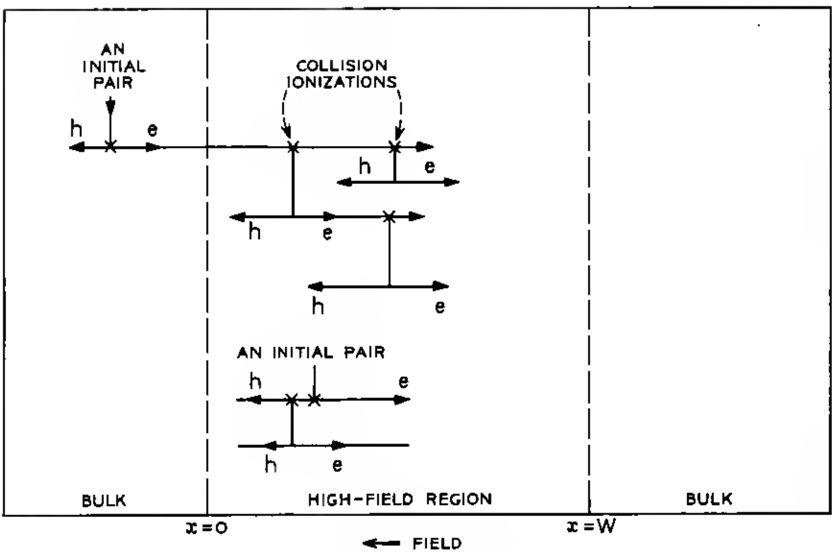

Fig. 1—Avalanche detector.

upon the position x) of suffering a collision ionization as they travel to the left under the influence of the electron field (see Fig. 1). Electrons traveling to the right have a probability $\alpha(x)$ per unit length of collision ionization. Carriers can be created within the high-field region due to thermal effects or due to the presence of incident light. Carriers can also drift into the region if they are generated outside of the region. It is assumed that all collision ionizations are independent. This requires that the mean distance between ionizing collisions be large compared to the distance over which a carrier can randomize its momentum after a collision.[3] Hole-electron pairs created through collision ionization can in turn generate additional pairs by the same mechanism. This is the avalanche process.

III. THE STATISTICS

We seek the statistics of the random total number of hole-electron pairs which result ultimately through collision ionizations when an initial hole-electron pair is injected into the high field at some position x. Define $p_g(n, x)$ as the probability that n pairs ultimately result including the initially injected pair. The moment-generating function of the number of pairs $M_g(s)$ is therefore[4]

$$M_g(s, x) = \sum_{1}^{\infty} p_g(n, x)e^{sn}. \tag{1}$$

We shall derive $M_g(s, x)$. Before proceeding we must review some well known results which will be needed in that derivation.

If $\{x_i\}$ are random variables which are independent, then the moment-generating function of the sum of the $\{x_i\}$ is the product of the individual moment-generating functions.[4] The semi-invariant moment-generating function SIMGF of a random variable X having probability density $p_X(x)$ is the natural logarithm of the moment-generating function of X

$$\psi_X(s) \equiv \ln\,[M_X(s)] = \ln\left[\sum_{-\infty}^{\infty} p_X(x)e^{sx}\right]. \tag{2}$$

The SIMGF of a sum of n independent random variables is the sum of the individual SIMGF's.

We can now proceed to derive $M_g(s, x)$. Divide the high-field region into K intervals of width $dX = W/K$. See Fig. 2. Label these intervals $1, 2, 3, \cdots j, \cdots K$. If a hole-electron pair is injected into interval j, define, as above, the probability density of the total number of pairs ultimately resulting in the avalanche process (including the initial pair)

as $p_o(n, x)$ where x is taken as the center of interval j. The hole of the initial pair moves to the left and the electron to the right toward $x = 0$ or $x = W$ respectively. As they pass through their respective intervals, new pairs may be created in each through collision ionization. We shall assume that the interval width dX is sufficiently narrow so that the initial pair carriers create either one or no new pairs in each interval. If the initial hole or electron generates a new pair in some interval k, then that new pair will ultimately generate N_k pairs including itself through the avalanche process. Thus with each *interval* we can associate a number of pairs N_k. This number equals zero if the appropriate initial pair carrier suffers no collision ionizations in interval k. This number equals one or more if the appropriate initial pair carrier suffers a collision ionization in interval k. The total number of pairs ultimately generated through the avalanche process including the initial pair is one plus the sum of the $\{N_k\}$. Since collision ionizations are all independent, all the N_k are independent. Thus we have the SIMGF of the total number of pairs given by

$$\psi_o(s, x) = s + \sum_{k=1}^{W/dX} \psi_{N_k}(s) \tag{3}$$

where s is the SIMGF of the deterministic initial pair and $\psi_{N_k}(s)$ is the SIMGF of N_k.

The SIMGF of N_k is obtained as follows. The probability that

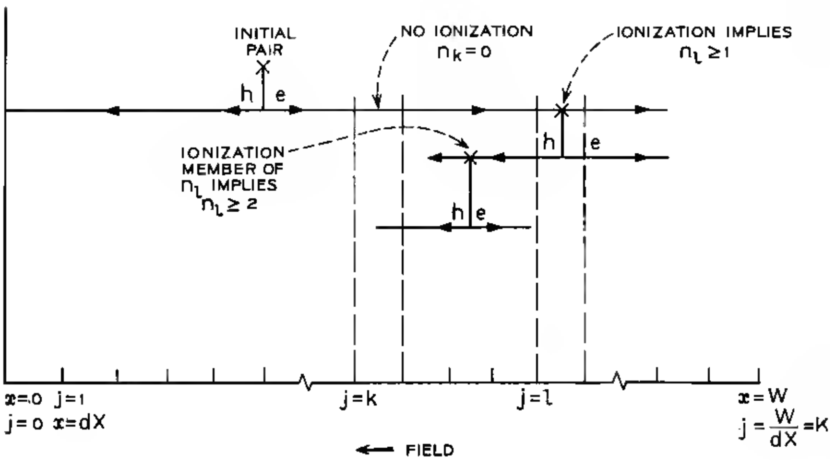

Fig. 2—Avalanche process.

$N_k = 0$ is $1 - \gamma\, dX$ where $\gamma = \alpha(x)$ if interval k is to the right of interval j where the initial pair enters, $\gamma = \beta(x)$ if k is to the left of j. (x is evaluated at the center of interval k.) The probability that $N_k = Z > 0$ is the probability of a collision ionization in interval k times the probability that a new pair created at interval k ultimately results in Z pairs including itself. That is

$$\Pr\,(N_k = Z) = \gamma\, dX\, p_g(Z, x) \quad \text{for} \quad Z > 0 \tag{4}$$

with x evaluated at the center of interval k. Thus

$$\begin{aligned} \psi_{N_k}(s) &= \ln\left[\sum_{Z=0}^{\infty} \Pr\,(N_k = Z)e^{sZ}\right] \\ &= \ln\left[(1 - \gamma\, dX)e^{s0} + \sum_{Z=1}^{\infty} \gamma\, dX\, p_g(Z, x)e^{sZ}\right] \\ &= \ln\,[1 - \gamma\, dX + \gamma\, dX\, M_g(s, x)] \\ &= \ln\,[1 - \gamma\, dX + \gamma\, dX\, e^{\psi_g(s,x)}] \end{aligned} \tag{5}$$

where

$$\begin{aligned} \gamma &= \alpha(x) \quad \text{if} \quad k > j \\ \gamma &= \beta(x) \quad \text{if} \quad k < j \\ x &= \text{center of interval } k. \end{aligned}$$

Using (3) and (5) and taking the limit as dX gets infinitely small, one obtains

$$\psi_g(s, x) = s + \int_0^x \beta(x')[e^{\psi_g(s,x')} - 1]\, dx' + \int_x^W \alpha(x')(e^{\psi_g(s,x')} - 1)\, dx'. \tag{6}$$

Equation (6) is the critical equation for determining $\psi_g(s, x)$, and thus $M_g(s, x) = \exp\,[\psi_g(s, x)]$. Using Leibnitz's rule for differentiation of integrals one obtains

$$\frac{\partial}{\partial x}\,\psi_g(s, x) = [\beta(x) - \alpha(x)](e^{\psi_g(s,x)} - 1). \tag{7}$$

The solution of (7) is

$$\psi_g(s, x) = \ln\left[\frac{1}{1 - C\exp\left(\int_0^x [\beta(x') - \alpha(x')]\, dx'\right)}\right] \tag{8}$$

where $C = (e^{\psi_g(s,0)} - 1)/e^{\psi_g(s,0)}$ which can be checked by substitution.

Substituting (8) into (6) one obtains the particular result

$$\psi_g(s, 0) = s + \int_0^W \alpha(x') \left[\frac{1}{1 - C \exp\left(\int_0^{x'} [\beta(x'') - \alpha(x'')]\, dx'' \right)} - 1 \right] dx'. \quad (9)$$

If one makes the *assumption*[3] that at each point in the high-field region

$$\beta(x) = k \cdot \alpha(x) \quad (10)$$

where k is a constant, one can solve (9) to obtain

$$\psi_g(s, 0) = s - \delta + \frac{1}{k-1} \ln \left[\frac{e^{(k-1)\delta}}{M_g(s, 0) - e^{(k-1)\delta}[M_g(s, 0) - 1]} \right]$$

$$= s + \frac{1}{1-k} \ln [M_g(s, 0) - e^{(k-1)\delta}[M_g(s, 0) - 1]] \quad (11)$$

where

$$M_g(s, 0) = e^{\psi_g(s,0)} \quad \text{and} \quad \delta = \int_0^W \alpha(x)\, dx.$$

One can write (11) in another way by making a substitution.* Define $\delta'(s)$ implicitly by

$$e^{\delta'(s)} = M_g(s, 0) e^{\delta - s}, \quad (12)$$

that is,

$$e^{\psi_g(s,0)} = M_g(s, 0) = e^{s - \delta + \delta'(s)}. \quad (13)$$

Using (12) in (11) one obtains the implicit equation

$$e^{-k\delta'(s)} - e^{-\delta'(s)} = e^s [e^{-k\delta} - e^{-\delta}] \quad (14)$$

which determines $\delta'(s)$ and thus $M_g(s, 0)$ through (13).

Equation (14) is still not explicit. A numerical technique for solution is discussed in the next section. One can use (11) [or (14)] and (8) to obtain $\psi_g(s, x)$ or $M_g(s, x)$ for any x. Recall that x is the point of entry of the initial pair. In the applications we shall be concerned with $x = 0$ or $x = W$. That is, pairs are generated in a drift region outside the high-field region with carriers drifting into the high-field region.

* Equation (14) will follow from (11) and (12) by tedious algebra. Further, (14) will not be used in the following results except to compare with McIntyre's work in Appendix A.

The equation (14) is consistent with some unpublished conjectures of McIntyre given in Appendix A.

IV. NUMERICAL SOLUTIONS

Equations (11) through (14) can be solved numerically. One technique is to differentiate (11) to obtain the result

$$\frac{\partial}{\partial s} M_g(s, 0) = M_g(s, 0)\left(\frac{k-1}{k}\right)\left[1 - \frac{1}{k}\left(M_g(s, 0)e^{-s}e^{\delta}\right)^{k-1}\right]^{-1} \tag{15}$$

where $M_g(0, 0) = 1$.

Equation (15) can be integrated with a computer to obtain $M_g(s, 0)$ explicitly.

V. SUMMARY OF ANALYTIC RESULTS ON AVALANCHE STATISTICS

From Sections I through IV and a previous paper by this author,[1] we obtained the following:

Assumptions:

Holes travel toward $x = 0$, electrons toward $x = W$.
Hole ionization probability per unit length $= \beta(x)$.
Electron ionization probability per unit length $= \alpha(x)$.
$\beta(x) = k \cdot \alpha(x)$, k a constant for all x.
High-field region width $= W$.

Definitions:

$$\delta \equiv \int_0^W \alpha(x)\, dx,$$

$p_g(n, x) =$ probability that if an initial pair enters tbe high-field region at point x, n pairs will ultimately result through the avalanche process including the initial pair,

$$\bar{G}(x) = \text{mean avalanche gain} = \sum_1^{\infty} n p_g(n, x),$$

$$M_g(s, x) \equiv \text{moment-generating function of } p_g(n, x)$$

$$= \sum_{n=1}^{\infty} p_g(n, x) e^{sn}.$$

Results:

$$1.\ M_g(s, x) = \left[1 - \frac{(M_g(s, 0) - 1)}{M_g(s, 0)} \exp\left[(k - 1)\int_0^x \alpha(x')\,dx'\right]\right]^{-1} \quad \text{for all} \quad k. \tag{16}$$

For $k = 0$ (Unilateral Gain)

$$2a.\ M_g(s, 0) = [1 - e^{\delta}[1 - e^{-s}]]^{-1} \quad \text{where} \quad \bar{G}(0) = e^{\delta}. \tag{17}$$

$$2b.\qquad p_g(n, 0) = \frac{1}{G}\left(\frac{G - 1}{G}\right)^{n-1} \quad \text{where} \quad G = \bar{G}(0). \tag{18}$$

For $k \neq 0$, $k \neq 1$

$$3.\ \frac{\partial}{\partial s} M_g(s, 0) = M_g(s, 0)\left(\frac{k - 1}{k}\right)\left[1 - \frac{1}{k}(M_g(s, 0)e^{-s}e^{\delta})^{k-1}\right]^{-1},$$

where

$$M_g(0, 0) = 1 \tag{19}$$

where

$$\bar{G}(0) = \left(\frac{k - 1}{k}\right)\left[1 - \frac{1}{k}e^{\delta(k-1)}\right]^{-1}.$$

For $k = 1$ (Equal Ionization)

$$4.\qquad \frac{\partial}{\partial s} M_g(s, 0) = M_g(s, 0)\left[1 - \left(\frac{G - 1}{G}\right)M_g(s, 0)\right]^{-1}$$

$$M_g(0, 0) = 1$$

where

$$G = \bar{G}(0). \tag{20}$$

VI. APPLICATIONS TO RECEIVERS USING AVALANCHE DETECTORS

6.1 *General Comments*

We shall next apply the results of Section V to obtain bounds on the error rates of digital receivers. The receivers to be discussed here are the single- and twin-channel systems described below. We shall upper-bound the power required at the receiver to obtain a desired error rate using the Chernov bounds.

6.2 *The Receivers*

The twin-channel receiver is shown in Fig. 3. Depending on the state of a binary information source, one of two channels has optical output

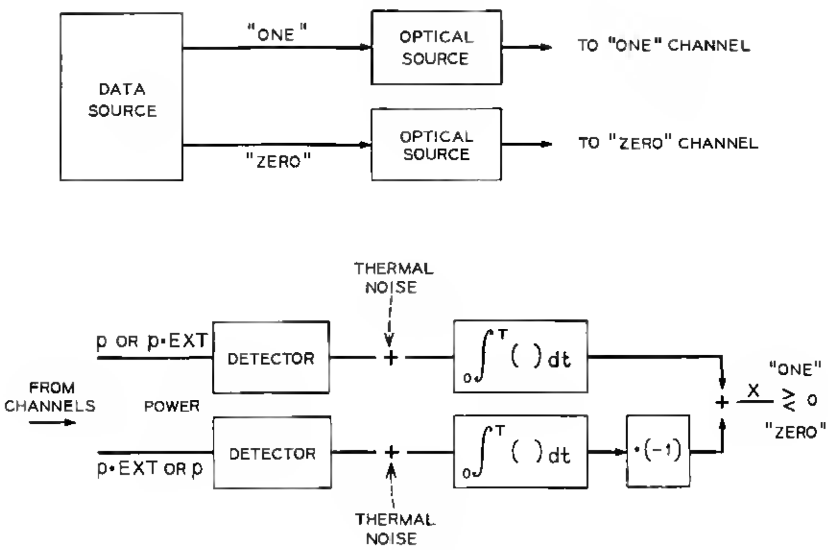

Fig. 3—Twin-channel system.

power p for duration T while the other has output power $p \cdot EXT$ for duration T. EXT signifies extinction. EXT would ideally be zero, but is left finite and less than unity for practical reasons. The optical power falling on an avalanche photo diode causes the emission of photoelectrons which are multiplied along with the detector "dark current" through the avalanche gain mechanism. The detector outputs (multiplied counts) are integrated in devices having thermal noises referred to their respective inputs. The integrator outputs are subtracted and the difference X is compared to a threshold of zero to decide what the information state was.

The single-channel system is essentially half the twin-channel system as shown in Fig. 4. The single integrator output X is compared to a threshold γ to decide upon the information state.

6.3 *The Chernov Bounds*

The Chernov bound is a useful tool for bounding the probability that a random variable X will lie above or below a given threshold γ. It is given as follows[5]

$$\Pr(X > \gamma) \leqq e^{\psi_X(s) - s\dot{\psi}_X(s)} \Big|_{\gamma = \psi'_X(s)}$$

provided $s > 0$,

$$\Pr(X < \gamma) \leqq e^{\psi_X(s) - s\psi_X'(s)} \Big|_{\gamma = \psi_X'(s)} \qquad \text{provided } s < 0, \qquad (21)$$

where

$$\psi_X(s) = \text{SIMGF of } X \equiv \ln\left[\int_{-\infty}^{\infty} p_X(x)e^{sx}\, dx\right]$$

and

$$\psi_X'(s) = \frac{\partial}{\partial s}\,\psi_X(s).$$

6.4 *Chernov Bounds for the Two Systems*

6.4.1 *Preliminaries*

We wish to determine the required power p to achieve a desired decision error probability for each of the receivers discussed in Section 6.2 with various types of detectors and various values of other system parameters such as dark current and thermal noise in the integrators. To apply the Chernov bounds we will need the SIMGF's of the variables X at the outputs of the receivers. (See Figs. 3 and 4.)

An important result needed here is the following:

Lemma: If C is integer-valued and greater than or equal to zero; and if $U = \sum_0^C g_i$ where the g_i are independent, identically distributed random variables (that is, each "count" produced by the C process independently

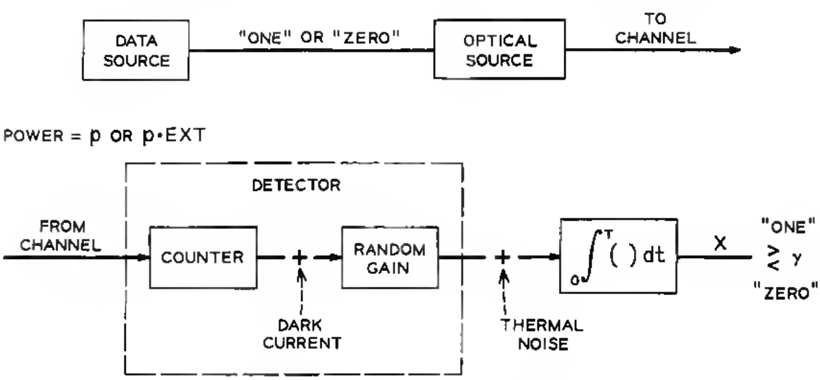

Fig. 4—Single-channel system.

generates g_i contributions to the U process through some gain mechanism); then the SIMGF of U is[1]

$$\psi_U(s) = \psi_C(\psi_g(s)) \tag{22}$$

where $\psi_g(s)$ is the SIMGF of the identically distributed random variables g_i.

One can model an avalanche photodiode as a photon counter followed by a random avalanche multiplier. Each photon counter output "count" produces a random number of "counts" at the multiplier output. Thus, since we know from Section V the SIMGF of a random multiplier which corresponds to $\psi_g(s)$ in (22), and since we seek the SIMGF of the photodiode output which corresponds to $\psi_U(s)$ in (22), it follows that we need the SIMGF of the number of counts emitted by a photon counter with light incident upon it, which corresponds to $\psi_C(s)$ in (22).

6.4.2 *Photon Counter Statistics*

6.4.2.1 *Coherent Light.* If the light incident upon the photon counter is coherent, then it is well known that the SIMGF of the total counts emitted in an interval T is given by[6] (see Appendix B)

$$\psi_C(s) = [LAMSIG + LAMD]\,[e^s - 1] \tag{23}$$

where

$LAMSIG$ = Total incident light energy $\cdot\, \eta/\hbar\Omega$
η = Detector quantum efficiency
$\hbar\Omega$ = Energy of photon at optical frequency used
$LAMD$ = Mean number of dark current counts before avalanche gain in an interval T.

6.4.2.2 *Incoherent Light.* If the incident light is incoherent with H independent spatial-temporal "degrees of freedom," then the SIMGF of the total counts emitted by the photon counter in interval T is (see Appendix B)

$$\psi_C = LAMD[e^s - 1] + \ln\left[\left[1 - \frac{LAMSIG}{H}(e^s - 1)\right]^{-H}\right] \tag{24}$$

where $LAMSIG$ is the average total incident energy times $\eta/\hbar\Omega$. Clearly (24) is the same as (23) as H approaches infinity, which is a well known result.

6.4.3 *Final Calculations*

6.4.3.1 *Twin-Channel System.* Since, for the twin-channel receiver, X consists of the difference of two random integrator outputs, we need

the following well known result.[4] If $X = X_1 - X_2$, and if X_1 and X_2 are independent, then the SIMGF of X is

$$\psi_X(s) = \psi_{X_1}(s) + \psi_{X_2}(-s). \tag{25}$$

Each integrator output contains the sum of the counts emitted by its detector and the integral of its thermal noise. The SIMGF of the random variable N obtained when Gaussian thermal noise spectral height N_0 is integrated over an interval T is well known to be

$$\psi_N(s) = \frac{s^2}{2} N_0 T. \tag{26}$$

Using (22) through (26) we obtain for the SIMGF of the twin-channel receiver output X when the information is in state "one", and the optical source is coherent,

$$\begin{aligned}\psi_X(s) = s^2 N_0 T &+ \left[\frac{p\cdot T\cdot \eta}{\hbar\Omega} + LAMD\right][M_g(s) - 1] \\ &+ \left[\frac{p\cdot T\cdot EXT\cdot \eta}{\hbar\Omega} + LAMD\right][M_g(-s) - 1]\end{aligned} \tag{27}$$

where

$LAMD$ = mean number of dark current counts before avalanche gain in an interval T

N_0 = spectral height of the thermal noises referred to the integrator inputs.

$M_g(s)$ is obtained from (16) through (20) depending upon the particular gain mechanism. If the optical source is incoherent, we have

$$\begin{aligned}\psi_X(s) = s^2 N_0 T &+ LAMD[M_g(s) + M_g(-s) - 2] \\ &+ \ln\left[\left[1 - \frac{p\cdot T\cdot \eta}{H\hbar\Omega}[M_g(s) - 1]\right]^{-H}\right] \\ &+ \ln\left[\left[1 - \frac{p\cdot T\cdot EXT\cdot \eta}{H\hbar\Omega}[M_g(-s) - 1]\right]^{-H}\right].\end{aligned} \tag{28}$$

We seek the probability that when the information is in state "one", X is less than zero, and we therefore decide that the information state was "zero." That is, we seek the error probability. One can use (27) or (28) and the Chernov bound of (21) to determine the required value of $LAMSIG \equiv p\cdot T\cdot \eta/(\hbar\Omega)$ to achieve a desired error probability. Since the twin-channel receiver is symmetric, the error probability

when the information is in state "zero" is the same as when it is in state "one."

6.4.3.2 *Single-Channel System.* For the single-channel receiver, we need the SIMGF of X under both information states. Call X_1 the random variable X when the information is in the state "one." Call X_0 the random variable X when the information is in state zero. Using results of Section 6.4, one obtains for coherent light

$$\psi_{X_1}(s) = \frac{s^2 N_0 T}{2} + \left[LAMD + \frac{p \cdot T \cdot \eta}{\hbar\Omega}\right][M_g(s) - 1]$$

$$\psi_{X_0}(s) = \frac{s^2 N_0}{2} + \left[LAMD + \frac{p \cdot T \cdot EXT \cdot \eta}{\hbar\Omega}\right][M_g(s) - 1]. \tag{29}$$

For incoherent light

$$\psi_{X_1}(s) = \frac{s^2 N_0 T}{2} + LAMD[M_g(s) - 1] + \ln\left[\left[1 - \frac{p \cdot T \cdot \eta}{H\hbar\Omega}[M_g(s) - 1]\right]^{-H}\right]$$

$$\psi_{X_0}(s) = \frac{s^2 N_0 T}{2} + LAMD[M_g(s) - 1] + \ln\left[\left[1 - \frac{p \cdot T \cdot EXT \cdot \eta}{H \cdot \hbar\Omega}[M_g(s) - 1]\right]^{-H}\right]. \tag{30}$$

One can then use the results of (29) and (30) along with the Chernov bounds of (21) to simultaneously find values of $LAMSIG = p \cdot T \cdot \eta/(\hbar\Omega)$ and the threshold γ (see Fig. 4) to ensure some desired error probability (which for convenience here will be the same for either information state).

6.5 *Numerical Results*

The Chernov bounds described above were evaluated numerically. The results are displayed on the attached figures described below. The range of parameter values is realistic and practical, to the best of this author's knowledge. The curves presented are those deemed most interesting by the author. Other calculations can of course be made. Parameters used are defined as follows:*

$LAMSIG$ = Required mean number of detected photons per pulse in the "on" channel of the twin-channel

* SIG, EXT, G, K, H, and $LAMD$ are input parameters to the program which calculates $LAMSIG$ for a desired error rate.

receiver or in the "one" state of the single-channel receiver.

$LAMSIG \cdot EXT$ = Mean number of detected counts per pulse in the "off" channel of the twin-channel receiver or in the "zero" state of the single-channel receiver.

SIG = Normalized thermal noise standard deviation

$$= \{4k\theta T/[\mathrm{Rc}^2]\}^{\frac{1}{2}} = \{4k\theta C/e^2\}^{\frac{1}{2}}$$

where e = electron charge, $k\theta$ = Boltzmann's constant·absolute temperature, R = equivalent noise resistance at integrator input, T = pulse duration, $C = T/R$ = integrator equivalent input capacitance. For the results to follow, a reasonable value of SIG was chosen to be 6000.

G = Mean avalanche gain.

H = Temporal-spatial diversity for incoherent carrier case.

k = Ratio of ionization probability per unit length of weaker and stronger ionizing carriers.*

$LAMD$ = Dark current counts per interval T before avalanche gain.

Fig. 5

$LAMSIG$ vs G is plotted for the twin-channel case with k as parameter. SIG was set at 6000, the error rate is 10^{-9}, $LAMD$ was set at 5 counts and $EXT = 0.01$. $H = 10{,}000$ which is equivalent to assuming a coherent carrier.

Fig. 6

The value at optimal gain of $LAMSIG$ vs k is plotted. Points are tagged with the optimal G. The receiver is a twin-channel system with $SIG = 6000$, $EXT = 0.01$, $LAMD = 5$. H is 10,000 which is equivalent to assuming a coherent carrier. The error rate is 10^{-9}.

Fig. 7

$LAMSIG$ vs G is plotted for two values of error rate 10^{-9} and 10^{-5} for

* For these calculations it was assumed that the detector is designed so that the stronger ionizing carriers generated optically or associated with dark current enter the high-field region from a drift region outside the high-field region. This corresponds to initial pairs entering the gain mechanism of $x = 0$ or $x = W$ as discussed in Section III.

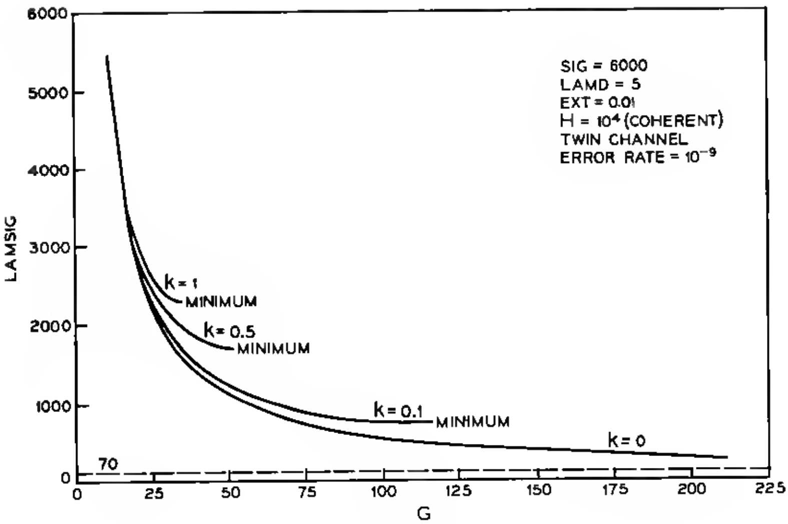

Fig. 5—LAMSIG versus gain.

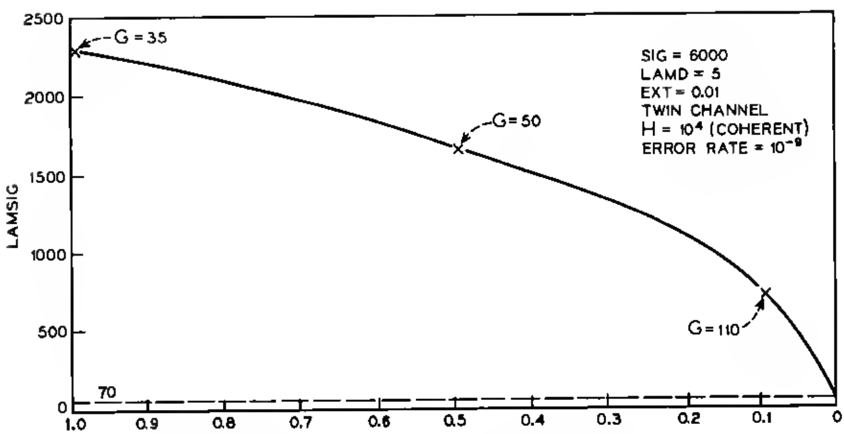

Fig. 6—LAMSIG at optimal gain versus k.

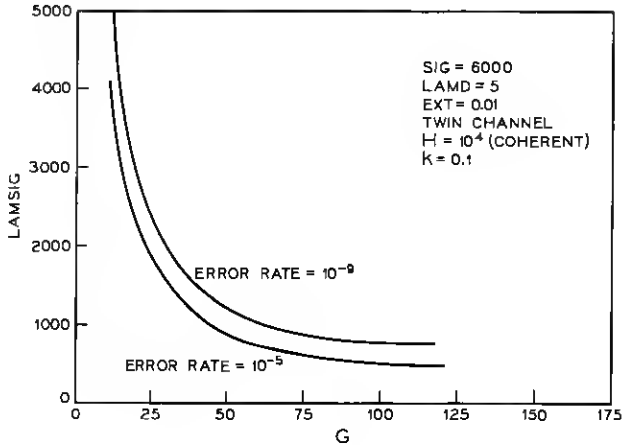

Fig. 7—LAMSIG versus gain.

a twin-channel system with $SIG = 6000$, $EXT = 0.01$, $H = 10{,}000$ $LAMD = 5$, $k = 0.1$.

Fig. 8

Single- and twin-channel systems are compared. $LAMSIG$ vs G is plotted for $SIG = 6000$, $EXT = 0.01$, $LAMD = 5$, $H = 10{,}000$, error rate $= 10^{-9}$, $k = 0$. Note that from an average power viewpoint the single-channel system is 3 dB better than shown if the binary information source is random, since $LAMSIG$ is the energy in "one" state.

Fig. 9

Same as Fig. 8 except $k = 1$. Note the scale change.

Fig. 10

$LAMSIG$ vs G for $H = 100$ and $H = 10{,}000$ for twin-channel system. $SIG = 6000$, $EXT = 0.01$, $LAMD = 5$, $k = 0$, error rate $= 10^{-9}$.

6.6 *Further Comments*

When systems were investigated for sensitivity to the choice $LAMD = 5$, $EXT = 0.01$, it was found that insignificant changes in $LAMSIG$ vs G occurred when various combinations of $LAMD = 5$ or 50, $EXT = 0.01$ or 0.001 were tried.

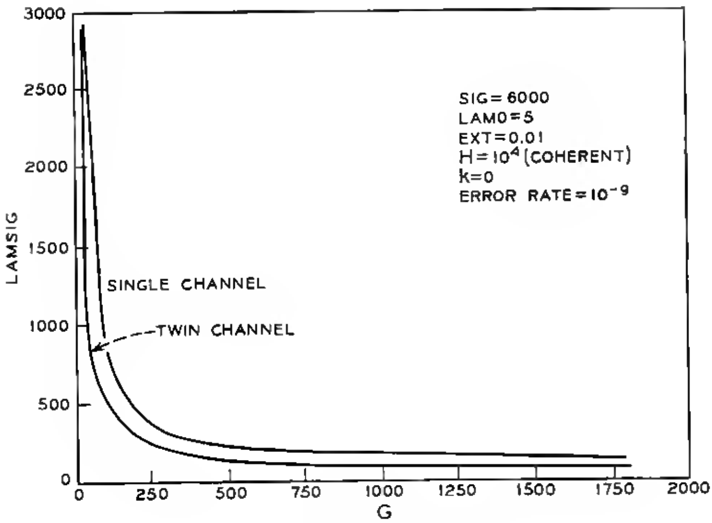

Fig. 8—LAMSIG versus gain.

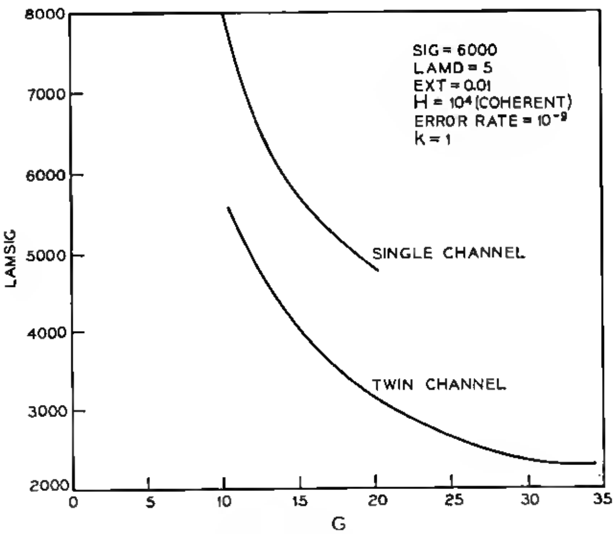

Fig. 9—LAMSIG versus gain.

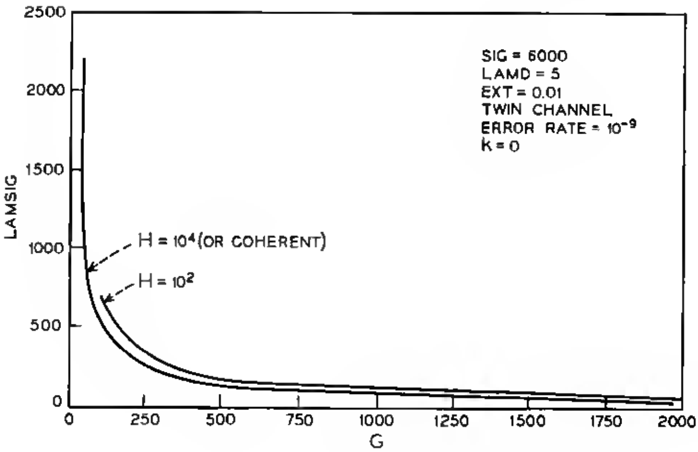

Fig. 10—LAMSIG versus gain.

VII. CONCLUSIONS ON APPLICATIONS

If one assumes that the Chernov bounds are sufficiently tight so that actual energy required per bit to achieve specified error rates can be compared for various system parameters by comparing the bounds,* then one can conclude the following.

(*i*) Define k as the ratio of the collision ionization probabilities per unit length of the weaker-ionizing to the stronger-ionizing carrier (carriers are of course holes and electrons). Assume that the detector is designed "well" such that optically and thermally generated carriers enter the high-field region from a drift region outside. From the bounds, one obtains the result that the required energy per pulse to achieve a desired error rate decreases as k decreases for fixed average avalanche gain. A value $k = 0$ is best; but a value $k = 0.1$ will allow one to operate with energy within 10 dB of that required at very high gains with a $k = 0$ device. For each value of k except zero there is an optimal gain resulting in minimum required energy per pulse. The optimal gain is larger for smaller k. At $k = 0.1$, the optimal gain is about 100. At $k = 0$, the optimal gain is infinite, but a gain of a few thousand allows

* For simple cases where both the bounds and actual energy requirements can be obtained (for instance for the $k = 0$ case) the two results differ by a few dB or less.

close to optimal required energy per pulse. One can conclude that a silicon device with $k = 0.1$ and a gain of about 100 would be a good choice for an optical detector. This is true since a detector with k less than 0.1 and yet having gain significantly higher than 100 is not available at this time.

(*ii*) The required energy per pulse for systems using incoherent optical sources differs from that for systems using coherent sources by less than a few dB provided the product of the source bandwidth and the pulse duration exceeds 100. This is true even if there is no spatial incoherence of the light at the detector.

(*iii*) For reasonable parameter values, and assuming a random information stream, the single-channel receiver requires about 1.5 dB less energy per pulse to achieve a desired error rate than the twin-channel receiver.

(*iv*) The required energy per pulse is insensitive to reasonable values of dark current and extinction ratios.

(*v*) For a particular system, a change in the desired error rate from 10^{-9} to 10^{-5} results in a change in the required energy per pulse of 1 to 3 dB, depending upon the avalanche gain. This shows that the required energy per pulse is fairly insensitive to the error rate. On the other hand, this means that poor error rates will result if insufficient loss margin is provided. That is, a small lowering of the received energy can greatly increase the error rate.

APPENDIX A

In an unpublished work, McIntyre conjectures (from special case calculations) that the probability density of the random gain, defined here as $p_g(n, 0)$ is given by

$$p_g(n, 0) = \frac{\Gamma\left(\frac{n}{1-k} + 1\right) e^{-\delta}(e^{-k\delta} - e^{-\delta})^{n-1}}{n!\, \Gamma\left(\frac{n}{1-k} + 2 - n\right)}$$

where k and δ are the same as in (10) through (13).

If one makes the assumption that the conjectured $p_g(n, 0)$ has sum over n normalized to unity for each value of k and for each δ, then one obtains the result of (14) by using the definition of the moment-generating function and the normalization property.

APPENDIX B

If light of *known* intensity falls upon a photon counter during an interval T, then the probability density of the total number of counts emitted is well known[6] to be Poisson distributed as follows

$$p_C(n) = [\Lambda + LAMD]^n \frac{e^{-(LAMD+\Lambda)}}{n!}. \tag{31}$$

Where Λ* is the total energy incident in the interval T times $\eta/\hbar\Omega$, $LAMD$ is the mean number of dark current counts per second times the interval T, and $\eta/\hbar\Omega$ is the detector quantum efficiency divided by the energy in a photon.

The moment-generating function of the distribution of (31) is given by

$$M_C(s) = \exp [[\Lambda + LAMD][e^s - 1]]. \tag{32}$$

If the incident light is a stochastic process, then the moment-generating function of the output count distribution is obtained by averaging (32) over the probability density of the stochastic total energy incident in the interval T

$$M_C(s) = \int_0^\infty \exp ([\Lambda + LAMD][e^s - 1])p(\Lambda)\, d\Lambda. \tag{33}$$

An incoherent light field is normally taken to mean that the complex envelope of the classical field is a complex Gaussian random process. That is, such a field incident on the photon counter plane can be written as

$$\begin{aligned} E(\rho, t) &= \sqrt{2}\,\mathrm{re}\{\epsilon(\rho, t)e^{i\Omega t}\} \\ &\rho \in \text{counter plane} \\ &t \in (0, T) \end{aligned} \tag{34}$$

where $\epsilon(\rho, t)$ is a complex Gaussian random process.

If one expands $\epsilon(\rho, t)$ in its Karhunen–Loeve eigenfunctions,[5] one obtains

$$\begin{aligned} \epsilon(\rho, t) &= \sum e_k\phi_k(\rho, t) \\ &\rho \in \text{counter plane} \\ &t \in (0, T) \end{aligned} \tag{35}$$

* In the text, Λ is called $LAMSIG$.

where

$$\int_{\text{counter plane}} \int_0^T \phi_k(\rho, t)\phi_j^*(\rho, t)\, d^2\rho\, dt = \delta_{k,j}$$

and the coefficients e_k are independent complex Gaussian random variables satisfying

$$\begin{aligned}\langle e_k e_j^* \rangle &= \gamma_k \delta_{k,j} \\ \langle e_k e_j \rangle &= 0.\end{aligned} \tag{36}$$

The energy incident upon the photon counter is

$$\frac{\hbar\Omega}{\eta}\Lambda = \int \epsilon(\rho, t)\epsilon^*(\rho, t)\, d^2\rho\, dt = \sum |e_k|^2. \tag{37}$$

If one assumes an equal distribution of average energy in roughly H "modes,"

$$\begin{aligned}\gamma_k &= \gamma, \qquad 1 \leqq k \leqq H \\ &= 0, \qquad k > H\end{aligned} \tag{38}$$

then it follows that from (33) and the complex Gaussian statistics of the e_k that

$$M_C(s) = \exp\left[LAMD(e^s - 1)\right]\cdot\left[1 - \frac{\eta}{\hbar\Omega}\gamma(e^s - 1)\right]^{-H}. \tag{39}$$

Assumption (38) implies that the energy of the incoherent light is roughly equally distributed in H degrees of freedom.

REFERENCES

1. Personick, S. D., "New Results on Avalanche Multiplication Statistics with Applications to Optical Detection," BSTJ, *50*, No. 1 (January 1971), pp. 167–189.
2. McIntyre, R. J., unpublished work (see Appendix A).
3. McIntyre, R. J., "Multiplication Noise in Uniform Avalanche Diodes," IEEE Trans. Elec. Dev., *ED-13*, No. 1 (January 1966), pp. 164–168.
4. Davenport, W., and Root, W., *Random Signals and Noise*, New York: John Wiley & Sons, 1958.
5. Van Trees, H. L., *Detection Estimation and Modulation, I*, New York: John Wiley & Sons, 1967.
6. Glauber, R. J., *Quantum Optics and Electronics, Les Honches 1964*, Lecture Notes at Session of the Summer School of Theoretical Physics at the Univ. of Grenoble, Ed. De Witt, New York: Gordon and Breach, 1965, p. 64.